Chapter 11: Chi-Square and ANOVA Tests

This chapter presents material on three more hypothesis tests. One is used to determine significant relationship between two qualitative variables, the second is used to determine if the sample data has a particular distribution, and the last is used to determine significant relationships between means of 3 or more samples.

Section 11.1: Chi-Square Test for Independence

Remember, qualitative data is where you collect data on individuals that are categories or names. Then you would count how many of the individuals had particular qualities. An example is that there is a theory that there is a relationship between breastfeeding and autism. To determine if there is a relationship, researchers could collect the time period that a mother breastfed her child and if that child was diagnosed with autism. Then you would have a table containing this information. Now you want to know if each cell is independent of each other cell. Remember, independence says that one event does not affect another event. Here it means that having autism is independent of being breastfed. What you really want is to see if they are not independent. In other words, does one affect the other? If you were to do a hypothesis test, this is your alternative hypothesis and the null hypothesis is that they are independent. There is a hypothesis test for this and it is called the **Chi-Square Test for Independence**. Technically it should be called the Chi-Square Test for Dependence, but for historical reasons it is known as the test for independence. Just as with previous hypothesis tests, all the steps are the same except for the assumptions and the test statistic.

Hypothesis Test for Chi-Square Test

1. State the null and alternative hypotheses and the level of significance
   H_o : the two variables are independent (this means that the one variable is not affected by the other)
   H_A : the two variables are dependent (this means that the one variable is affected by the other)
   Also, state your α level here.

2. State and check the assumptions for the hypothesis test
   a. A random sample is taken.
   b. Expected frequencies for each cell are greater than or equal to 5 (The expected frequencies, *E*, will be calculated later, and this assumption means $E \geq 5$).

3. Find the test statistic and p-value
   Finding the test statistic involves several steps. First the data is collected and counted, and then it is organized into a table (in a table each entry is called a cell). These values are known as the observed frequencies, which the symbol for an observed frequency is *O*. Each table is made up of rows and columns. Then each row is totaled to give a row total and each column is totaled to give a column total.

The null hypothesis is that the variables are independent. Using the multiplication rule for independent events you can calculate the probability of being one value of the first variable, A, and one value of the second variable, B (the probability of a particular cell $P(A \text{ and } B)$). Remember in a hypothesis test, you assume that H_0 is true, the two variables are assumed to be independent.

$$
\begin{aligned}
P(A \text{ and } B) &= P(A)\cdot P(B) \text{ if A and B are independent} \\
&= \frac{\text{number of ways A can happen}}{\text{total number of individuals}}\cdot\frac{\text{number of ways B can happen}}{\text{total number of individuals}} \\
&= \frac{\text{row total}}{n} * \frac{\text{column total}}{n}
\end{aligned}
$$

Now you want to find out how many individuals you expect to be in a certain cell. To find the expected frequencies, you just need to multiply the probability of that cell times the total number of individuals. Do not round the expected frequencies.

$$
\begin{aligned}
\text{Expected frequency}(\text{cell } A \text{ and } B) &= E(A \text{ and } B) \\
&= n\left(\frac{\text{row total}}{n}\cdot\frac{\text{column total}}{n}\right) \\
&= \frac{\text{row total}\cdot\text{column total}}{n}
\end{aligned}
$$

If the variables are independent the expected frequencies and the observed frequencies should be the same. The test statistic here will involve looking at the difference between the expected frequency and the observed frequency for each cell. Then you want to find the “total difference” of all of these differences. The larger the total, the smaller the chances that you could find that test statistic given that the assumption of independence is true. That means that the assumption of independence is not true. How do you find the test statistic? First find the differences between the observed and expected frequencies. Because some of these differences will be positive and some will be negative, you need to square these differences. These squares could be large just because the frequencies are large, you need to divide by the expected frequencies to scale them. Then finally add up all of these fractional values. This is the test statistic.

Test Statistic:

The symbol for Chi-Square is χ^2

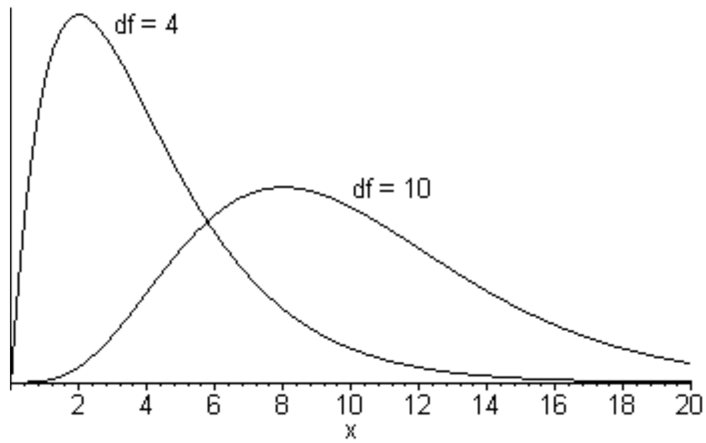
$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

where O is the observed frequency and E is the expected frequency

Distribution of Chi-Square

χ^2 has different curves depending on the degrees of freedom. It is skewed to the right for small degrees of freedom and gets more symmetric as the degrees of freedom increases (see figure #11.1.1). Since the test statistic involves squaring the differences, the test statistics are all positive. A chi-squared test for independence is always right tailed.

Figure #11.1.1: Chi-Square Distribution



p-value:

Use $\chi\text{cdf}(\text{lower limit,1E99, } df)$

Where the degrees of freedom is $df = (\#\text{ of rows} - 1) * (\#\text{ of columns} - 1)$

4. Conclusion
   This is where you write reject H_o or fail to reject H_o. The rule is: if the p-value < α, then reject H_o. If the p-value $\geq \alpha$, then fail to reject H_o

5. Interpretation
   This is where you interpret in real world terms the conclusion to the test. The conclusion for a hypothesis test is that you either have enough evidence to show H_A is true, or you do not have enough evidence to show H_A is true.

Example #11.1.1: Hypothesis Test with Chi-Square Test Using Formula

Is there a relationship between autism and breastfeeding? To determine if there is, a researcher asked mothers of autistic and non-autistic children to say what time period they breastfed their children. The data is in table #11.1.1 (Schultz, Klonoff-Cohen, Wingard, Askhoomoff, Macera, Ji & Bacher, 2006). Do the data provide enough evidence to show that that breastfeeding and autism are independent? Test at the1% level.

Table #11.1.1: Autism Versus Breastfeeding

Autism	Breast Feeding Timelines				Row Total
	None	Less than 2 months	2 to 6 months	More than 6 months	
Yes	241	198	164	215	818
No	20	25	27	44	116
Column Total	261	223	191	259	934

Solution:

1. State the null and alternative hypotheses and the level of significance
   H_o : Breastfeeding and autism are independent
   H_A : Breastfeeding and autism are dependent
   $\alpha = 0.01$

2. State and check the assumptions for the hypothesis test
   a. A random sample of breastfeeding time frames and autism incidence was taken.
   b. Expected frequencies for each cell are greater than or equal to 5 (ie. $E \geq 5$). See step 3. All expected frequencies are more than 5.

3. Find the test statistic and p-value
   Test statistic:
   First find the expected frequencies for each cell.
   $E(\text{Autism and no breastfeeding}) = \dfrac{818*261}{934} \approx 228.585$
   $E(\text{Autism and} < 2\text{ months}) = \dfrac{818*223}{934} \approx 195.304$
   $E(\text{Autism and 2 to 6 months}) = \dfrac{818*191}{934} \approx 167.278$
   $E(\text{Autism and more than 6 months}) = \dfrac{818*259}{934} \approx 226.833$
   Others are done similarly. It is easier to do the calculations for the test statistic with a table, the others are in table #11.1.2 along with the calculation for the test statistic. (Note: the column of $O-E$ should add to 0 or close to 0.)

Table #11.1.2: Calculations for Chi-Square Test Statistic

O	E	$O-E$	$(O-E)^2$	$(O-E)^2/E$
241	228.585	12.415	154.132225	0.674288448
198	195.304	2.696	7.268416	0.03721591
164	167.278	-3.278	10.745284	0.064236086
215	226.833	-11.833	140.019889	0.617281828
20	32.4154	-12.4154	154.1421572	4.755213792
25	27.6959	-2.6959	7.26787681	0.262417066
27	23.7216	3.2784	10.74790656	0.453085229
44	32.167	11.833	140.019889	4.352904809
Total		0.0001		$11.2166432 = \chi^2$

The test statistic formula is $\chi^2 = \sum \frac{(O-E)^2}{E}$, which is the total of the last column in table #11.1.2.

p-value:

$df = (2-1)*(4-1) = 3$

$\chi\text{cdf}(11.2166432, 1E99, 3) \approx 0.01061$

4. Conclusion
   Fail to reject H_o since the p-value is more than 0.01.

5. Interpretation
   There is not enough evidence to show that breastfeeding and autism are dependent. This means that you cannot say that the whether a child is breastfed or not will indicate if that the child will be diagnosed with autism.

Example #11.1.2: Hypothesis Test with Chi-Square Test Using TI-83/84 Calculator

Is there a relationship between autism and breastfeeding? To determine if there is, a researcher asked mothers of autistic and non-autistic children to say what time period they breastfed their children. The data is in table #11.1.1 (Schultz, Klonoff-Cohen, Wingard, Askhoomoff, Macera, Ji & Bacher, 2006). Do the data provide enough evidence to show that that breastfeeding and autism are independent? Test at the1% level.

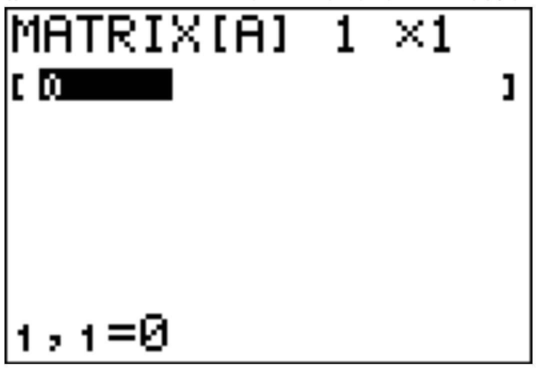
Solution:

1. State the null and alternative hypotheses and the level of significance
   H_o : Breastfeeding and autism are independent
   H_A : Breastfeeding and autism are dependent
   $\alpha = 0.01$

2. State and check the assumptions for the hypothesis test
   a. A random sample of breastfeeding time frames and autism incidence was taken.
   b. Expected frequencies for each cell are greater than or equal to 5 (ie. $E \geq 5$). See step 3. All expected frequencies are more than 5.

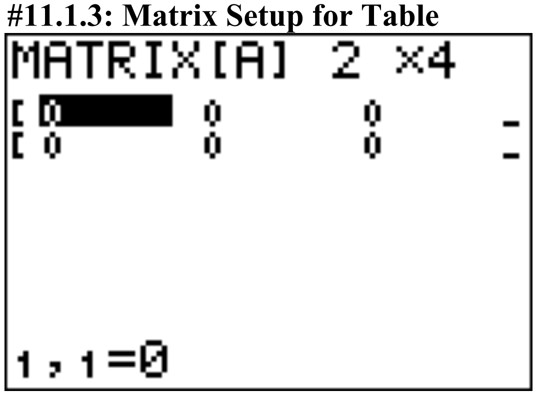
3. Find the test statistic and p-value
   Test statistic:
   To use the calculator to compute the test statistic, you must first put the data into the calculator. However, this process is different than for other hypothesis tests. You need to put the data in as a matrix instead of in the list. Go into the MATRX menu then move over to EDIT and choose 1:[A]. This will allow you to type the table into the calculator. Figure #11.1.2 shows what you will see on your calculator when you choose 1:[A] from the EDIT menu.

   Figure #11.1.2: Matrix Edit Menu on TI-83/84



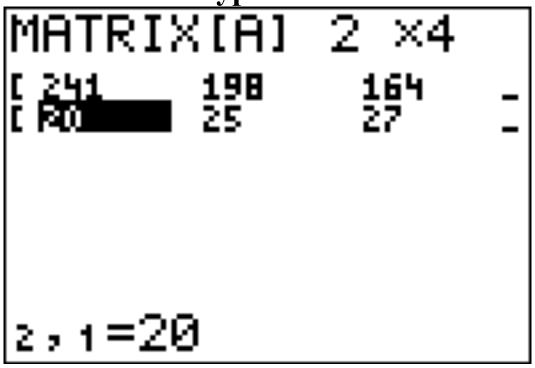
   The table has 2 rows and 4 columns (don't include the row total column and the column total row in your count). You need to tell the calculator that you have a 2 by 4. The 1 X1 (you might have another size in your matrix, but it doesn't matter because you will change it) on the calculator is the size of the matrix. So type 2 ENTER and 4 ENTER and the calculator will make a matrix of the correct size. See figure #11.1.3.

   Figure #11.1.3: Matrix Setup for Table



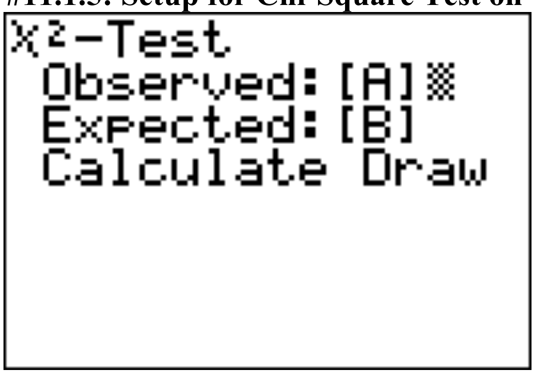
Now type the table in by pressing ENTER after each cell value. Figure #11.1.4 contains the complete table typed in. Once you have the data in, press QUIT.

Figure #11.1.4: Data Typed into Matrix



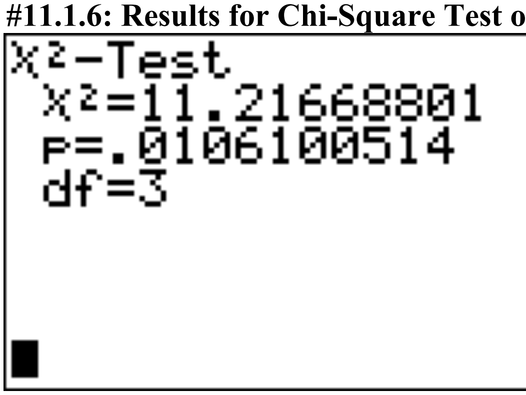
To run the test on the calculator, go into STAT, then move over to TEST and choose χ^2-Test from the list. The setup for the test is in figure #11.1.5.

Figure #11.1.5: Setup for Chi-Square Test on TI-83/84



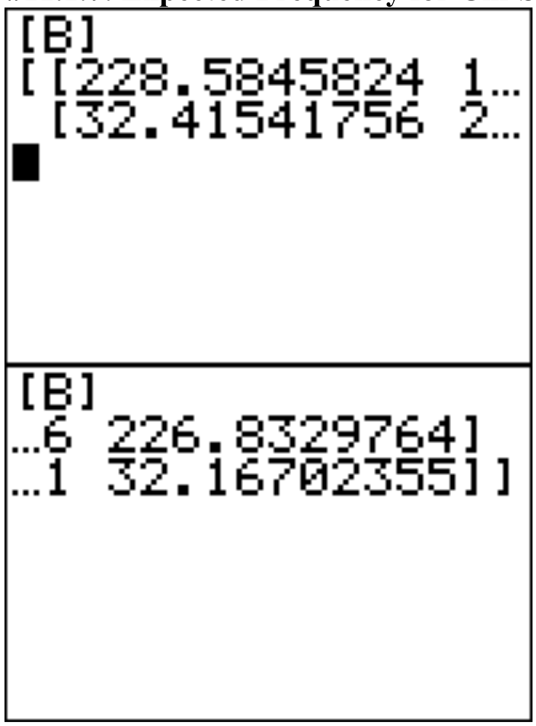
Once you press ENTER on Calculate you will see the results in figure #11.1.6.

Figure #11.1.6: Results for Chi-Square Test on TI-83/84



The test statistic is $\chi^2 \approx 11.2167$ and the p-value is $p \approx 0.01061$. Notice that the calculator calculates the expected values for you and places them in matrix B. To review the expected values, go into MATRX and choose 2:[B]. Figure #11.1.7 shows the output. Press the right arrows to see the entire matrix.

Figure #11.1.7: Expected Frequency for Chi-Square Test on TI-83/84



4. Conclusion
   Fail to reject H_o since the p-value is more than 0.01.

5. Interpretation
   There is not enough evidence to show that breastfeeding and autism are dependent. This means that you cannot say that the whether a child is breastfed or not will indicate if that the child will be diagnosed with autism.

Example #11.1.3: Hypothesis Test with Chi-Square Test Using Formula

The World Health Organization (WHO) keeps track of how many incidents of leprosy there are in the world. Using the WHO regions and the World Banks income groups, one can ask if an income level and a WHO region are dependent on each other in terms of predicting where the disease is. Data on leprosy cases in different countries was collected for the year 2011 and a summary is presented in table #11.1.3 ("Leprosy: Number of," 2013). Is there evidence to show that income level and WHO region are independent when dealing with the disease of leprosy? Test at the 5% level.

Table #11.1.3: Number of Leprosy Cases

WHO Region	World Bank Income Group				Row Total
	High Income	Upper Middle Income	Lower Middle Income	Low Income	
Americas	174	36028	615	0	36817
Eastern Mediterranean	54	6	1883	604	2547
Europe	10	0	0	0	10
Western Pacific	26	216	3689	1155	5086
Africa	0	39	1986	15928	17953
South-East Asia	0	0	149896	10236	160132
Column Total	264	36289	158069	27923	222545

Solution:

1. State the null and alternative hypotheses and the level of significance
   H_o : WHO region and Income Level when dealing with the disease of leprosy are independent
   H_A : WHO region and Income Level when dealing with the disease of leprosy are dependent
   $\alpha = 0.05$

2. State and check the assumptions for the hypothesis test
   a. A random sample of incidence of leprosy was taken from different countries and the income level and WHO region was taken.
   b. Expected frequencies for each cell are greater than or equal to 5 (ie. $E \geq 5$). See step 3. There are actually 4 expected frequencies that are less than 5, and the results of the test may not be valid. If you look at the expected frequencies you will notice that they are all in Europe. This is because Europe didn't have many cases in 2011.

3. Find the test statistic and p-value
   Test statistic:
   First find the expected frequencies for each cell.

$$E(\text{Americas and High Income}) = \frac{36817 * 264}{222545} \approx 43.675$$

$$E(\text{Americas and Upper Middle Income}) = \frac{36817 * 36289}{222545} \approx 6003.514$$

$$E(\text{Americas and Lower Middle Income}) = \frac{36817 * 158069}{222545} \approx 26150.335$$

$$E(\text{Americas and Lower Income}) = \frac{36817 * 27923}{222545} \approx 4619.475$$

Others are done similarly. It is easier to do the calculations for the test statistic with a table, and the others are in table #11.1.4 along with the calculation for the test statistic.

Table #11.1.4: Calculations for Chi-Square Test Statistic

O	E	$O-E$	$(O-E)^2$	$(O-E)^2/E$
174	43.675	130.325	16984.564	388.8838719
54	3.021	50.979	2598.813	860.1218328
10	0.012	9.988	99.763	8409.746711
26	6.033	19.967	398.665	66.07628214
0	21.297	-21.297	453.572	21.29722977
0	189.961	-189.961	36085.143	189.9608978
36028	6003.514	30024.486	901469735.315	150157.0038
6	415.323	-409.323	167545.414	403.4097962
0	1.631	-1.631	2.659	1.6306365
216	829.342	-613.342	376188.071	453.5983897
39	2927.482	-2888.482	8343326.585	2850.001268
0	26111.708	-26111.708	681821316.065	26111.70841
615	26150.335	-25535.335	652053349.724	24934.7988
1883	1809.080	73.920	5464.144	3.020398811
0	7.103	-7.103	50.450	7.1027882
3689	3612.478	76.522	5855.604	1.620938405
1986	12751.636	-10765.636	115898911.071	9088.944681
149896	113738.368	36157.632	1307374351.380	11494.57632
0	4619.475	-4619.475	21339550.402	4619.475122
604	319.575	284.425	80897.421	253.1404187
0	1.255	-1.255	1.574	1.25471253
1155	638.147	516.853	267137.238	418.6140882
15928	2252.585	13675.415	187016964.340	83023.25138
10236	20091.963	-9855.963	97140000.472	4834.769106
Total		0.000		$328594.008 = \chi^2$

The test statistic formula is $\chi^2 = \sum \frac{(O-E)^2}{E}$, which is the total of the last column in table #11.1.2.

p-value:

$$df = (6-1)*(4-1) = 15$$

$$\chi\text{cdf}(328594.008, 1E99, 15) \approx 0$$

4. Conclusion
   Reject H_o since the p-value is less than 0.05.

5. Interpretation
   There is enough evidence to show that WHO region and income level are dependent when dealing with the disease of leprosy. WHO can decide how to focus their efforts based on region and income level. Do remember though that the results may not be valid due to the expected frequencies not all be more than 5.

Example #11.1.4: Hypothesis Test with Chi-Square Test Using TI-83/84 Calculator

The World Health Organization (WHO) keeps track of how many incidents of leprosy there are in the world. Using the WHO regions and the World Banks income groups, one can ask if an income level and a WHO region are dependent on each other in terms of predicting where the disease is. Data on leprosy cases in different countries was collected for the year 2011 and a summary is presented in table #11.1.3 ("Leprosy: Number of," 2013). Is there evidence to show that income level and WHO region are independent when dealing with the disease of leprosy? Test at the 5% level.

Solution:

1. State the null and alternative hypotheses and the level of significance
   H_o : WHO region and Income Level when dealing with the disease of leprosy are independent
   H_A : WHO region and Income Level when dealing with the disease of leprosy are dependent
   $\alpha = 0.05$

2. State and check the assumptions for the hypothesis test
   a. A random sample of incidence of leprosy was taken from different countries and the income level and WHO region was taken.
   b. Expected frequencies for each cell are greater than or equal to 5 (ie. $E \geq 5$). See step 3. There are actually 4 expected frequencies that are less than 5, and the results of the test may not be valid. If you look at the expected frequencies you will notice that they are all in Europe. This is because Europe didn't have many cases in 2011.

3. Find the test statistic and p-value
   Test statistic:
   See example #11.1.2 for the process of doing the test on the calculator. Remember, you need to put the data in as a matrix instead of in the list.

Figure #11.1.8: Setup for Matrix on TI-83/84

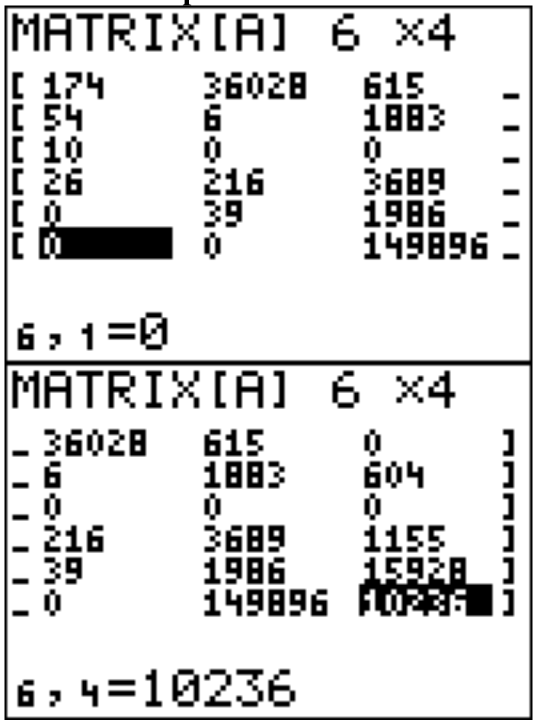
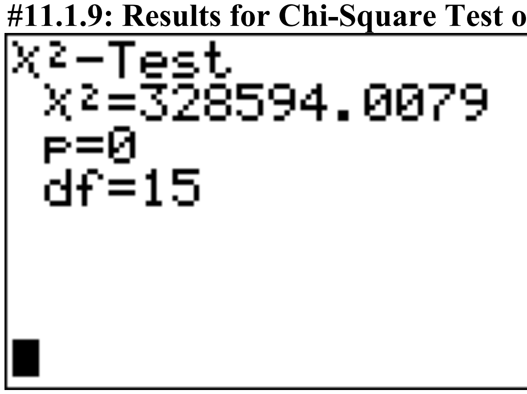
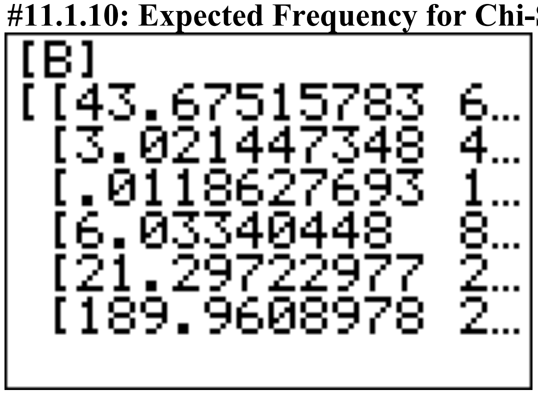


Figure #11.1.9: Results for Chi-Square Test on TI-83/84



$\chi^2 \approx 328594.0079$

Figure #11.1.10: Expected Frequency for Chi-Square Test on TI-83/84



Press the right arrow to look at the other expected frequencies.

p-value:

$$p-value \approx 0$$

4. Conclusion
   Reject H_o since the p-value is less than 0.05.

5. Interpretation
   There is enough evidence to show that WHO region and income level are dependent when dealing with the disease of leprosy. WHO can decide how to focus their efforts based on region and income level. Do remember though that the results may not be valid due to the expected frequencies not all be more than 5.

Section 11.1: Homework

In each problem show all steps of the hypothesis test. If some of the assumptions are not met, note that the results of the test may not be correct and then continue the process of the hypothesis test.

1.) The number of people who survived the Titanic based on class and sex is in table #11.1.5 ("Encyclopedia Titanica," 2013). Is there enough evidence to show that the class and the sex of a person who survived the Titanic are independent? Test at the 5% level.

Table #11.1.5: Surviving the Titanic

	Sex		
Class	Female	Male	Total
1st	134	59	193
2nd	94	25	119
3rd	80	58	138
Total	308	142	450

2.) Researchers watched groups of dolphins off the coast of Ireland in 1998 to determine what activities the dolphins partake in at certain times of the day ("Activities of dolphin," 2013). The numbers in table #11.1.6 represent the number of groups of dolphins that were partaking in an activity at certain times of days. Is there enough evidence to show that the activity and the time period are independent for dolphins? Test at the 1% level.

Table #11.1.6: Dolphin Activity

	Period				Row
Activity	Morning	Noon	Afternoon	Evening	Total
Travel	6	6	14	13	39
Feed	28	4	0	56	88
Social	38	5	9	10	62
Column Total	72	15	23	79	189

3.) Is there a relationship between autism and what an infant is fed? To determine if there is, a researcher asked mothers of autistic and non-autistic children to say what they fed their infant. The data is in table #11.1.7 (Schultz, Klonoff-Cohen, Wingard, Askhoomoff, Macera, Ji & Bacher, 2006). Do the data provide enough evidence to show that that what an infant is fed and autism are independent? Test at the1% level.

Table #11.1.7: Autism Versus Breastfeeding

Autism	Feeding: Brest-feeding	Feeding: Formula with DHA/ARA	Feeding: Formula without DHA/ARA	Row Total
Yes	12	39	65	116
No	6	22	10	38
Column Total	18	61	75	154

4.) A person's educational attainment and age group was collected by the U.S. Census Bureau in 1984 to see if age group and educational attainment are related. The counts in thousands are in table #11.1.8 ("Education by age," 2013). Do the data show that educational attainment and age are independent? Test at the 5% level.

Table #11.1.8: Educational Attainment and Age Group

Education	Age Group: 25-34	35-44	45-54	55-64	>64	Row Total
Did not complete HS	5416	5030	5777	7606	13746	37575
Competed HS	16431	1855	9435	8795	7558	44074
College 1-3 years	8555	5576	3124	2524	2503	22282
College 4 or more years	9771	7596	3904	3109	2483	26863
Column Total	40173	20057	22240	22034	26290	130794

5.) Students at multiple grade schools were asked what their personal goal (get good grades, be popular, be good at sports) was and how important good grades were to them (1 very important and 4 least important). The data is in table #11.1.9 ("Popular kids datafile," 2013). Do the data provide enough evidence to show that goal attainment and importance of grades are independent? Test at the 5% level.

Table #11.1.9: Personal Goal and Importance of Grades

Goal	Grades Importance Rating 1	2	3	4	Row Total
Grades	70	66	55	56	247
Popular	14	33	45	49	141
Sports	10	24	33	23	90
Column Total	94	123	133	128	478

6.) Students at multiple grade schools were asked what their personal goal (get good grades, be popular, be good at sports) was and how important being good at sports were to them (1 very important and 4 least important). The data is in table #11.1.10 ("Popular kids datafile," 2013). Do the data provide enough evidence to show that goal attainment and importance of sports are independent? Test at the 5% level.

Table #11.1.10: Personal Goal and Importance of Sports

Goal	Sports Importance Rating 1	2	3	4	Row Total
Grades	83	81	55	28	247
Popular	32	49	43	17	141
Sports	50	24	14	2	90
Column Total	165	154	112	47	478

7.) Students at multiple grade schools were asked what their personal goal (get good grades, be popular, be good at sports) was and how important having good looks were to them (1 very important and 4 least important). The data is in table #11.1.11 ("Popular kids datafile," 2013). Do the data provide enough evidence to show that goal attainment and importance of looks are independent? Test at the 5% level.

Table #11.1.11: Personal Goal and Importance of Looks

Goal	Looks Importance Rating 1	2	3	4	Row Total
Grades	80	66	66	35	247
Popular	81	30	18	12	141
Sports	24	30	17	19	90
Column Total	185	126	101	66	478

8.) Students at multiple grade schools were asked what their personal goal (get good grades, be popular, be good at sports) was and how important having money were to them (1 very important and 4 least important). The data is in table #11.1.12 ("Popular kids datafile," 2013). Do the data provide enough evidence to show that goal attainment and importance of money are independent? Test at the 5% level.

Table #11.1.12: Personal Goal and Importance of Money

	Money Importance Rating				
Goal	1	2	3	4	Row Total
Grades	14	34	71	128	247
Popular	14	29	35	63	141
Sports	6	12	26	46	90
Column Total	34	75	132	237	478

Section 11.2: Chi-Square Goodness of Fit

In probability, you calculated probabilities using both experimental and theoretical methods. There are times when it is important to determine how well the experimental values match the theoretical values. An example of this is if you wish to verify if a die is fair. To determine if observed values fit the expected values, you want to see if the difference between observed values and expected values is large enough to say that the test statistic is unlikely to happen if you assume that the observed values fit the expected values. The test statistic in this case is also the chi-square. The process is the same as for the chi-square test for independence.

Hypothesis Test for Goodness of Fit Test

1. State the null and alternative hypotheses and the level of significance
   H_o : The data are consistent with a specific distribution
   H_A : The data are not consistent with a specific distribution
   Also, state your α level here.

2. State and check the assumptions for the hypothesis test
   a. A random sample is taken.
   b. Expected frequencies for each cell are greater than or equal to 5 (The expected frequencies, *E*, will be calculated later, and this assumption means $E \geq 5$).

3. Find the test statistic and p-value
   Finding the test statistic involves several steps. First the data is collected and counted, and then it is organized into a table (in a table each entry is called a cell). These values are known as the observed frequencies, which the symbol for an observed frequency is *O*. The table is made up of *k* entries. The total number of observed frequencies is *n*. The expected frequencies are calculated by multiplying the probability of each entry, *p*, times *n*.

$$\text{Expected frequency}\left(\text{entry } i\right) = E = n * p$$

Test Statistic:

$$\chi^2 = \sum \frac{\left(O - E\right)^2}{E}$$

where *O* is the observed frequency and *E* is the expected frequency

Again, the test statistic involves squaring the differences, so the test statistics are all positive. Thus a chi-squared test for goodness of fit is always right tailed.

p-value:

Use $\chi\text{cdf}\left(\text{lower limit,1E99, } df\right)$

Where the degrees of freedom is $df = k - 1$

4. Conclusion
   This is where you write reject H_o or fail to reject H_o. The rule is: if the p-value $< \alpha$, then reject H_o. If the p-value $\geq \alpha$, then fail to reject H_o

5. Interpretation
   This is where you interpret in real world terms the conclusion to the test. The conclusion for a hypothesis test is that you either have enough evidence to show H_A is true, or you do not have enough evidence to show H_A is true.

Example #11.2.1: Goodness of Fit Test Using the Formula

Suppose you have a die that you are curious if it is fair or not. If it is fair then the proportion for each value should be the same. You need to find the observed frequencies and to accomplish this you roll the die 500 times and count how often each side comes up. The data is in table #11.2.1. Do the data show that the die is fair? Test at the 5% level.

Table #11.2.1: Observed Frequencies of Die

Die values	1	2	3	4	5	6	Total
Observed Frequency	78	87	87	76	85	87	100

Solution:

1. State the null and alternative hypotheses and the level of significance
   H_o: The observed frequencies are consistent with the distribution for fair die (the die is fair)
   H_A: The observed frequencies are not consistent with the distribution for fair die (the die is not fair)
   $\alpha = 0.05$

2. State and check the assumptions for the hypothesis test
   a. A random sample is taken since each throw of a die is a random event.
   b. Expected frequencies for each cell are greater than or equal to 5. See step 3.

3. Find the test statistic and p-value
   First you need to find the probability of rolling each side of the die. The sample space for rolling a die is $\{1, 2, 3, 4, 5, 6\}$. Since you are assuming that the die is fair, then $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$.
   Now you can find the expected frequency for each side of the die. Since all the probabilities are the same, then each expected frequency is the same.

$$\text{Expected frequency} = E = n * p = 500 * \frac{1}{6} \approx 83.33$$

Test Statistic:
It is easier to calculate the test statistic using a table.

Table #11.2.2: Calculation of the Chi-Square Test Statistic

O	E	$O-E$	$(O-E)^2$	$\frac{(O-E)^2}{E}$
78	83.33	-5.33	28.4089	0.340920437
87	83.33	3.67	13.4689	0.161633265
87	83.33	3.67	13.4689	0.161633265
76	83.33	-7.33	53.7289	0.644772591
85	83.33	1.67	2.7889	0.033468139
87	83.33	3.67	13.4689	0.161633265
Total		0.02		$\chi^2 \approx$ 1.504060962

The test statistic is $\chi^2 \approx 1.504060962$

The degrees of freedom are $df = k-1 = 6-1 = 5$

The $p-value = \chi^2\text{cdf}(1.50406096, 1E99, 5) \approx 0.913$

4. Conclusion
   Fail to reject H_o since the p-value is greater than 0.05.

5. Interpretation
   There is not enough evidence to show that the die is not consistent with the distribution for a fair die. There is not enough evidence to show that the die is not fair.

Example #11.2.2: Goodness of Fit Test Using the TI-84 Calculator

Suppose you have a die that you are curious if it is fair or not. If it is fair then the proportion for each value should be the same. You need to find the observed frequencies and to accomplish this you roll the die 500 times and count how often each side comes up. The data is in table #11.2.1. Do the data show that the die is fair? Test at the 5% level.

Solution:

1. State the null and alternative hypotheses and the level of significance
   H_o : The observed frequencies are consistent with the distribution for fair die (the die is fair)
   H_A : The observed frequencies are not consistent with the distribution for fair die (the die is not fair)
   $\alpha = 0.05$

2. State and check the assumptions for the hypothesis test
   a. A random sample is taken since each throw of a die is a random event.
   b. Expected frequencies for each cell are greater than or equal to 5. See step 3.

3. Find the test statistic and p-value
   TI-83:
   To use the TI-83 calculator to compute the test statistic, you must first put the data into the calculator. Type the observed frequencies into L1 and the expected frequencies into L2. Then you will need to go to L3, arrow up onto the name, and type in $(L1 - L2)^2 / L2$. Now you use 1-Var Stats L3 to find the total. See figure #11.2.1 for the initial setup, figure #11.2.2 for the results of that calculation, and figure #11.2.3 for the result of the 1-Var Stats L3.

Figure #11.2.1: Input into TI-83

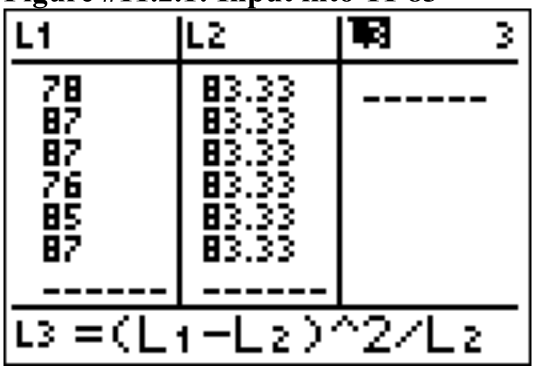


Figure #11.2.2: Result for L3 on TI-83

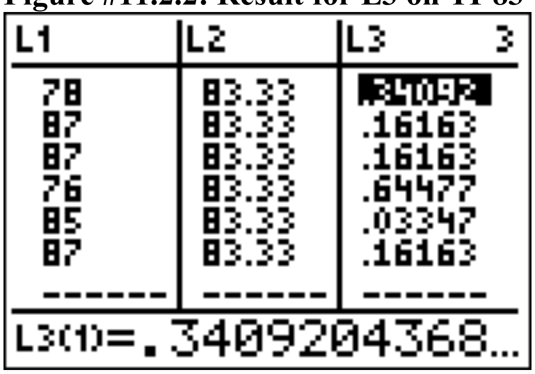
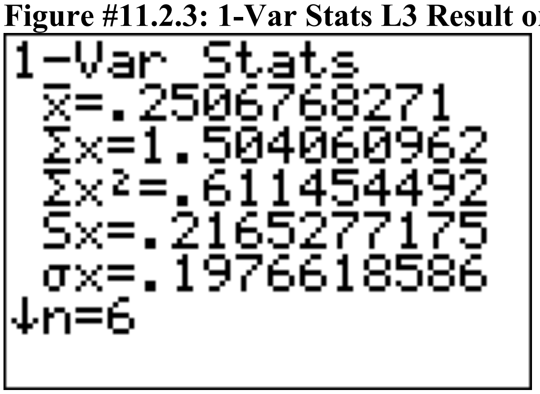


Figure #11.2.3: 1-Var Stats L3 Result on TI-83

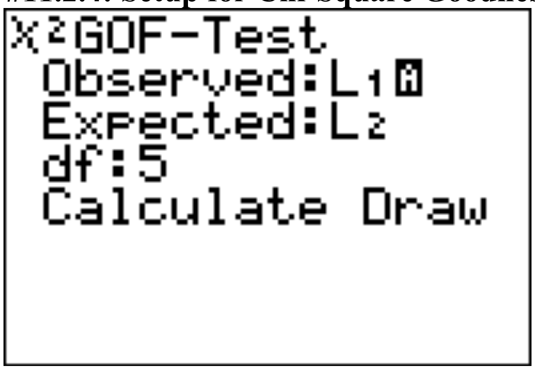


The total is the chi-square value, $\chi^2 = \sum x \approx 1.50406$.

The p-value is found using $p-value = \chi^2\text{cdf}\left(1.50406096, 1E99, 5\right) \approx 0.913$, where the degrees of freedom is $df = k-1 = 6-1 = 5$.

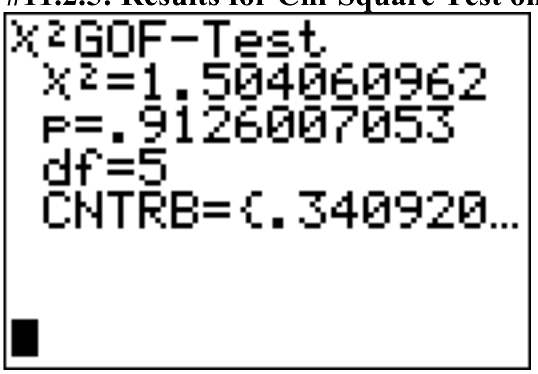
TI-84:
To run the test on the TI-84, type the observed frequencies into L1 and the expected frequencies into L2, then go into STAT, move over to TEST and choose χ^2 GOF-Test from the list. The setup for the test is in figure #11.2.4.

Figure #11.2.4: Setup for Chi-Square Goodness of Fit Test on TI-84



Once you press ENTER on Calculate you will see the results in figure #11.2.5.

Figure #11.2.5: Results for Chi-Square Test on TI-83/84



The test statistic is $\chi^2 \approx$ 1.504060962
The $p-value \approx 0.913$
The CNTRB represent the $\frac{(O-E)^2}{E}$ for each die value. You can see the values by pressing the right arrow.

4. Conclusion
Fail to reject H_o since the p-value is greater than 0.05.

5. Interpretation
   There is not enough evidence to show that the die is not consistent with the distribution for a fair die. There is not enough evidence to show that the die is not fair.

Section 11.2: Homework

In each problem show all steps of the hypothesis test. If some of the assumptions are not met, note that the results of the test may not be correct and then continue the process of the hypothesis test.

1.) According to the M&M candy company, the expected proportion can be found in Table #11.2.3. In addition, the table contains the number of M&M's of each color that were found in a case of candy (Madison, 2013). At the 5% level, do the observed frequencies support the claim of M&M?

Table #11.2.3: M&M Observed and Proportions

	Blue	Brown	Green	Orange	Red	Yellow	Total
Observed Frequencies	481	371	483	544	372	369	2620
Expected Proportion	0.24	0.13	0.16	0.20	0.13	0.14	

2.) Eyeglassomatic manufactures eyeglasses for different retailers. They test to see how many defective lenses they made the time period of January 1 to March 31. Table #11.2.4 gives the defect and the number of defects.

Table #11.2.4: Number of Defective Lenses

Defect type	Number of defects
Scratch	5865
Right shaped – small	4613
Flaked	1992
Wrong axis	1838
Chamfer wrong	1596
Crazing, cracks	1546
Wrong shape	1485
Wrong PD	1398
Spots and bubbles	1371
Wrong height	1130
Right shape – big	1105
Lost in lab	976
Spots/bubble – intern	976

Do the data support the notion that each defect type occurs in the same proportion? Test at the 10% level.

3.) On occasion, medical studies need to model the proportion of the population that has a disease and compare that to observed frequencies of the disease actually occurring. Suppose the end-stage renal failure in south-west Wales was collected for different age groups. Do the data in table 11.2.5 show that the observed frequencies are in agreement with proportion of people in each age group (Boyle, Flowerdew & Williams, 1997)? Test at the 1% level.

Table #11.2.5: Renal Failure Frequencies

Age Group	16-29	30-44	45-59	60-75	75+	Total
Observed Frequency	32	66	132	218	91	539
Expected Proportion	0.23	0.25	0.22	0.21	0.09	

4.) In Africa in 2011, the number of deaths of a female from cardiovascular disease for different age groups are in table #11.2.6 ("Global health observatory," 2013). In addition, the proportion of deaths of females from all causes for the same age groups are also in table #11.2.6. Do the data show that the death from cardiovascular disease are in the same proportion as all deaths for the different age groups? Test at the 5% level.

Table #11.2.6: Deaths of Females for Different Age Groups

Age	5-14	15-29	30-49	50-69	Total
Cardiovascular Frequency	8	16	56	433	513
All Cause Proportion	0.10	0.12	0.26	0.52	

5.) In Australia in 1995, there was a question of whether indigenous people are more likely to die in prison than non-indigenous people. To figure out, the data in table 11.2.7 was collected. ("Aboriginal deaths in," 2013). Do the data show that indigenous people die in the same proportion as non-indigenous people? Test at the 1% level.

Table #11.2.7: Death of Prisoners

	Prisoner Dies	Prisoner Did Not Die	Total
Indigenous Prisoner Frequency	17	2890	2907
Frequency of Non-Indigenous Prisoner	42	14459	14501

6.) A project conducted by the Australian Federal Office of Road Safety asked people many questions about their cars. One question was the reason that a person chooses a given car, and that data is in table #11.2.8 ("Car preferences," 2013).

Table #11.2.8: Reason for Choosing a Car

Safety	Reliability	Cost	Performance	Comfort	Looks
84	62	46	34	47	27

Do the data show that the frequencies observed substantiate the claim that the reasons for choosing a car are equally likely? Test at the 5% level.

Section 11.3: Analysis of Variance (ANOVA)

There are times where you want to compare three or more population means. One idea is to just test different combinations of two means. The problem with that is that your chance for a type I error increases. Instead you need a process for analyzing all of them at the same time. This process is known as **analysis of variance (ANOVA)**. The test statistic for the ANOVA is fairly complicated, you will want to use technology to find the test statistic and p-value. The test statistic is distributed as an F-distribution, which is skewed right and depends on degrees of freedom. Since you will use technology to find these, the distribution and the test statistic will not be presented. Remember, all hypothesis tests are the same process. Note that to obtain a statistically significant result there need only be a difference between any two of the *k* means.

Before conducting the hypothesis test, it is helpful to look at the means and standard deviations for each data set. If the sample means with consideration of the sample standard deviations are different, it may mean that some of the population means are different. However, do realize that if they are different, it doesn't provide enough evidence to show the population means are different. Calculating the sample statistics just gives you an idea that conducting the hypothesis test is a good idea.

Hypothesis test using ANOVA to compare *k* means

1. State the random variables and the parameters in words

   x_1 = random variable 1

   x_2 = random variable 2

   $\vdots$

   x_k = random variable k

   μ_1 = mean of random variable 1

   μ_2 = mean of random variable 2

   $\vdots$

   μ_k = mean of random variable k

2. State the null and alternative hypotheses and the level of significance

   $H_o : \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_k$

   H_A : at least two of the means are not equal

   Also, state your α level here.

3. State and check the assumptions for the hypothesis test
   a. A random sample of size n_i is taken from each population.
   b. All the samples are independent of each other.
   c. Each population is normally distributed. The ANOVA test is fairly robust to the assumption especially if the sample sizes are fairly close to each other. Unless the populations are really not normally distributed and the sample sizes are close to each other, then this is a loose assumption.

d. The population variances are all equal. If the sample sizes are close to each other, then this is a loose assumption.

4. Find the test statistic and p-value

   The test statistic is $F=\dfrac{MS_B}{MS_W}$, where $MS_B=\dfrac{SS_B}{df_B}$ is the mean square between the groups (or factors), and $MS_W=\dfrac{SS_W}{df_W}$ is the mean square within the groups. The degrees of freedom between the groups is $df_B=k-1$ and the degrees of freedom within the groups is $df_W=n_1+n_2+\cdots+n_k-k$. To find all of the values, use technology such as the TI-83/84 calculator.

   The test statistic, F, is distributed as an F-distribution, where both degrees of freedom are needed in this distribution. The p-value is also calculated by the calculator.

5. Conclusion

   This is where you write reject H_o or fail to reject H_o. The rule is: if the p-value < α, then reject H_o. If the p-value $\geq \alpha$, then fail to reject H_o

6. Interpretation

   This is where you interpret in real world terms the conclusion to the test. The conclusion for a hypothesis test is that you either have enough evidence to show H_A is true, or you do not have enough evidence to show H_A is true.

If you do in fact reject H_o, then you know that at least two of the means are different. The next question you might ask is which are different? You can look at the sample means, but realize that these only give a preliminary result. To actually determine which means are different, you need to conduct other tests. Some of these tests are the range test, multiple comparison tests, Duncan test, Student-Newman-Keuls test, Tukey test, Scheffé test, Dunnett test, least significant different test, and the Bonferroni test. There is no consensus on which test to use. These tests are available in statistical computer packages such as Minitab and SPSS.

Example #11.3.1: Hypothesis Test Involving Several Means

Cancer is a terrible disease. Surviving may depend on the type of cancer the person has. To see if the mean survival time for several types of cancer are different, data was collected on the survival time in days of patients with one of these cancer in advanced stage. The data is in table #11.3.1 ("Cancer survival story," 2013). (Please realize that this data is from 1978. There have been many advances in cancer treatment, so do not use this data as an indication of survival rates from these cancers.) Do the data indicate that at least two of the mean survival time for these types of cancer are not all equal? Test at the 1% level.

Table #11.3.1: Survival Times in Days of Five Cancer Types

Stomach	Bronchus	Colon	Ovary	Breast
124	81	248	1234	1235
42	461	377	89	24
25	20	189	201	1581
45	450	1843	356	1166
412	246	180	2970	40
51	166	537	456	727
1112	63	519		3808
46	64	455		791
103	155	406		1804
876	859	365		3460
146	151	942		719
340	166	776		
396	37	372		
	223	163		
	138	101		
	72	20		
	245	283		

Solution:

1. State the random variables and the parameters in words

$x_1 =$ survival time from stomach cancer
$x_2 =$ survival time from bronchus cancer
$x_3 =$ survival time from colon cancer
$x_4 =$ survival time from ovarian cancer
$x_5 =$ survival time from breast cancer
$\mu_1 =$ mean survival time from stomach cancer
$\mu_2 =$ mean survival time from bronchus cancer
$\mu_3 =$ mean survival time from colon cancer
$\mu_4 =$ mean survival time from ovarian cancer
$\mu_5 =$ mean survival time from breast cancer

Now before conducting the hypothesis test, look at the means and standard deviations.

$\bar{x}_1 = 286$ $\quad s_1 \approx 346.31$
$\bar{x}_2 \approx 211.59$ $\quad s_2 \approx 209.86$
$\bar{x}_3 \approx 457.41$ $\quad s_3 \approx 427.17$
$\bar{x}_4 \approx 884.33$ $\quad s_4 \approx 1098.58$
$\bar{x}_5 \approx 1395.91$ $\quad s_5 \approx 1238.97$

There appears to be a difference between at least two of the means, but realize that the standard deviations are very different. The difference you see may not be significant.

Notice the sample sizes are not the same. The sample sizes are
$n_1 = 13, n_2 = 17, n_3 = 17, n_4 = 6, n_5 = 11$

2. State the null and alternative hypotheses and the level of significance
   $H_o : \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$
   H_A : at least two of the means are not equal
   $\alpha = 0.01$

3. State and check the assumptions for the hypothesis test
   a. A random sample of 13 survival times from stomach cancer was taken. A random sample of 17 survival times from bronchus cancer was taken. A random sample of 17 survival times from colon cancer was taken. A random sample of 6 survival times from ovarian cancer was taken. A random sample of 11 survival times from breast cancer was taken. These statements may not be true. This information was not shared as to whether the samples were random or not but it may be safe to assume that.
   b. Since the individuals have different cancers, then the samples are independent.
   c. Population of all survival times from stomach cancer is normally distributed. Population of all survival times from bronchus cancer is normally distributed. Population of all survival times from colon cancer is normally distributed. Population of all survival times from ovarian cancer is normally distributed. Population of all survival times from breast cancer is normally distributed. Looking at the histograms and normal probability plots fore each sample, it appears that none of the populations are normally distributed. The sample sizes are somewhat different for the problem. This assumption may not be true.
   d. The population variances are all equal. The sample standard deviations are approximately 346.3, 209.9, 427.2, 1098.6, and 1239.0 respectively. This assumption does not appear to be met, since the sample standard deviations are very different. The sample sizes are somewhat different for the problem. This assumption may not be true.

4. Find the test statistic and p-value
   Type each data set into L1 through L5. Then go into STAT and over to TESTS and choose ANOVA(. Then type in L1,L2,L3,L4,L5 and press enter. You will get the results of the ANOVA test.

Figure #11.3.1: Setup for ANOVA on TI-83/84

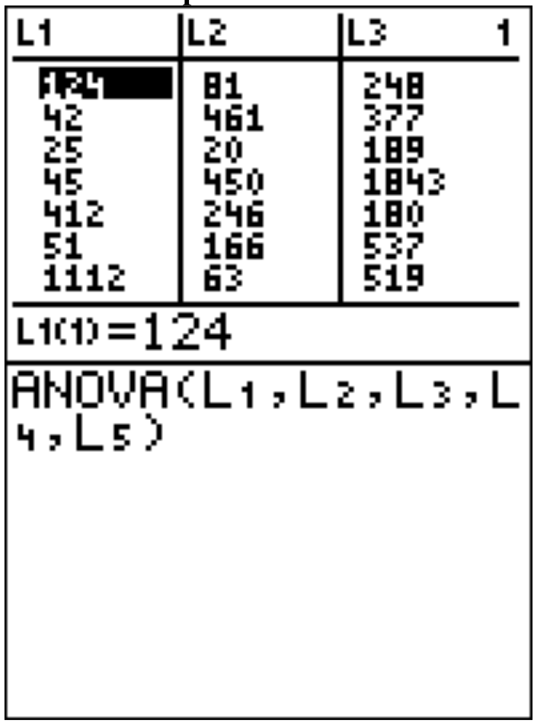
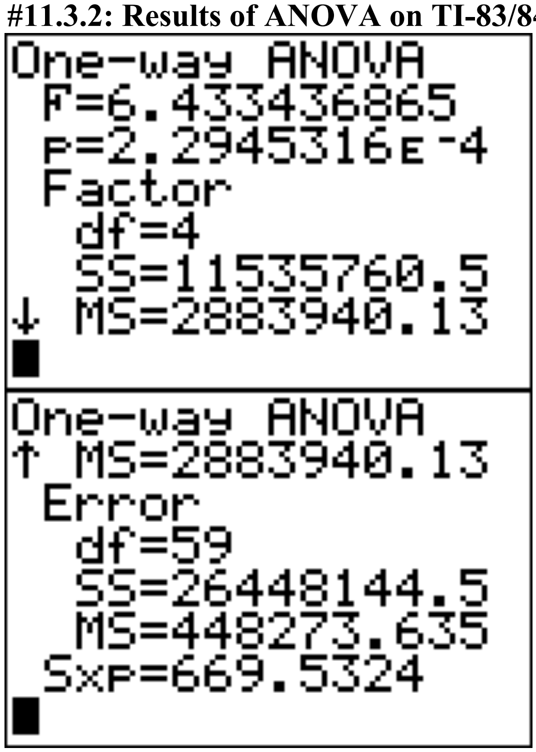


Figure #11.3.2: Results of ANOVA on TI-83/84



The test statistic is $F \approx 6.433$ and $p - value \approx 2.29 \times 10^{-4}$.

Just so you know, the Factor information is between the groups and the Error is within the groups. So

$MS_B \approx 2883940.13$, $SS_B \approx 11535760.5$, and $df_B = 4$ and
$MS_W \approx 448273.635$, $SS_W \approx 448273.635$, and $df_W = 59$.

5. Conclusion
   Reject H_o since the p-value is less than 0.01.

6. Interpretation
   There is evidence to show that at least two of the mean survival times from different cancers are not equal.

   By examination of the means, it appears that the mean survival time for breast cancer is different from the mean survival times for both stomach and bronchus cancers. It may also be different for the mean survival time for colon cancer. The others may not be different enough to actually say for sure.

Section 11.3: Homework

In each problem show all steps of the hypothesis test. If some of the assumptions are not met, note that the results of the test may not be correct and then continue the process of the hypothesis test.

1.) Cuckoo birds are in the habit of laying their eggs in other birds' nest. The other birds adopt and hatch the eggs. The lengths (in cm) of cuckoo birds' eggs in the other species nests were measured and are in table #11.3.2 ("Cuckoo eggs in," 2013). Do the data show that the mean length of cuckoo bird's eggs is not all the same when put into different nests? Test at the 5% level.

Table #11.3.2: Lengths of Cuckoo Bird Eggs in Different Species Nests

Meadow Pipit		Tree Pipit	Hedge Sparrow	Robin	Pied Wagtail	Wren
19.65	22.25	21.05	20.85	21.05	21.05	19.85
20.05	22.45	21.85	21.65	21.85	21.85	20.05
20.65	22.45	22.05	22.05	22.05	21.85	20.25
20.85	22.45	22.45	22.85	22.05	21.85	20.85
21.65	22.65	22.65	23.05	22.05	22.05	20.85
21.65	22.65	23.25	23.05	22.25	22.45	20.85
21.65	22.85	23.25	23.05	22.45	22.65	21.05
21.85	22.85	23.25	23.05	22.45	23.05	21.05
21.85	22.85	23.45	23.45	22.65	23.05	21.05
21.85	22.85	23.45	23.85	23.05	23.25	21.25
22.05	23.05	23.65	23.85	23.05	23.45	21.45
22.05	23.25	23.85	23.85	23.05	24.05	22.05
22.05	23.25	24.05	24.05	23.05	24.05	22.05
22.05	23.45	24.05	25.05	23.05	24.05	22.05
22.05	23.65	24.05		23.25	24.85	22.25
22.05	23.85			23.85		
22.05	24.25					
22.05	24.45					
22.05	22.25					
22.05	22.25					
22.25	22.25					
22.25	22.25					
22.25						

2.) Levi-Strauss Co manufactures clothing. The quality control department measures weekly values of different suppliers for the percentage difference of waste between the layout on the computer and the actual waste when the clothing is made (called run-up). The data is in table #11.3.3, and there are some negative values because sometimes the supplier is able to layout the pattern better than the computer ("Waste run up," 2013). Do the data show that there is a difference between some of the suppliers? Test at the 1% level.

Table #11.3.3: Run-ups for Different Plants Making Levi Strauss Clothing

Plant 1	Plant 2	Plant 3	Plant 4	Plant 5
1.2	16.4	12.1	11.5	24
10.1	-6	9.7	10.2	-3.7
-2	-11.6	7.4	3.8	8.2
1.5	-1.3	-2.1	8.3	9.2
-3	4	10.1	6.6	-9.3
-0.7	17	4.7	10.2	8
3.2	3.8	4.6	8.8	15.8
2.7	4.3	3.9	2.7	22.3
-3.2	10.4	3.6	5.1	3.1
-1.7	4.2	9.6	11.2	16.8
2.4	8.5	9.8	5.9	11.3
0.3	6.3	6.5	13	12.3
3.5	9	5.7	6.8	16.9
-0.8	7.1	5.1	14.5	
19.4	4.3	3.4	5.2	
2.8	19.7	-0.8	7.3	
13	3	-3.9	7.1	
42.7	7.6	0.9	3.4	
1.4	70.2	1.5	0.7	
3	8.5			
2.4	6			
1.3	2.9			

3.) Several magazines were grouped into three categories based on what level of education of their readers the magazines are geared towards: high, medium, or low level. Then random samples of the magazines were selected to determine the number of three-plus-syllable words were in the advertising copy, and the data is in table #11.3.4 ("Magazine ads readability," 2013). Is there enough evidence to show that the mean number of three-plus-syllable words in advertising copy is different for at least two of the education levels? Test at the 5% level.

Table #11.3.4: Number of Three Plus Syllable Words in Advertising Copy

High Education	Medium Education	Low Education
34	13	7
21	22	7
37	25	7
31	3	7
10	5	7
24	2	7
39	9	8
10	3	8
17	0	8
18	4	8
32	29	8
17	26	8
3	5	9
10	5	9
6	24	9
5	15	9
6	3	9
6	8	9

4.) A study was undertaken to see how accurate food labeling for calories on food that is considered reduced calorie. The group measured the amount of calories for each item of food and then found the percent difference between measured and labeled food, $\frac{(\text{measured} - \text{labeled})}{\text{labeled}} * 100\%$. The group also looked at food that was nationally advertised, regionally distributed, or locally prepared. The data is in table #11.3.5 ("Calories datafile," 2013). Do the data indicate that at least two of the mean percent differences between the three groups are different? Test at the 10% level.

Table #11.3.5: Percent Differences Between Measured and Labeled Food

National Advertised	Regionally Distributed	Locally Prepared
2	41	15
-28	46	60
-6	2	250
8	25	145
6	39	6
-1	16.5	80
10	17	95
13	28	3
15	-3	
-4	14	
-4	34	
-18	42	
10		
5		
3		
-7		
3		
-0.5		
-10		
6		

5.) The amount of sodium (in mg) in different types of hotdogs is in table #11.3.6 ("Hot dogs story," 2013). Is there sufficient evidence to show that the mean amount of sodium in the types of hotdogs are not all equal? Test at the 5% level.

Table #11.3.6: Amount of Sodium (in mg) in Beef, Meat, and Poultry Hotdogs

Beef	Meat	Poultry
495	458	430
477	506	375
425	473	396
322	545	383
482	496	387
587	360	542
370	387	359
322	386	357
479	507	528
375	393	513
330	405	426
300	372	513
386	144	358
401	511	581
645	405	588
440	428	522
317	339	545
319		
298		
253		

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